

ESTIMATES OF LIFESPAN AND BLOW-UP RATES FOR THE WAVE EQUATION WITH A TIME-DEPENDENT DAMPING AND A POWER-TYPE NONLINEARITY

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ABSTRACT. We study estimates of lifespan and blow-up rates of solutions for the Cauchy problem of the wave equation with a time-dependent damping and a power-type nonlinearity. When the damping acts on the solutions effectively, and the nonlinearity belongs to the subcritical case, we show the sharp lifespan estimates and the blow-up rates of solutions. The upper estimates are proved by an ODE argument, and the lower estimates are given by a method of scaling variables.

1. INTRODUCTION

We consider the Cauchy problem of the wave equation with a time-dependent damping and a power-type nonlinearity

$$\begin{cases} \square u + b(t)u_t = |u|^p, & t \in [0, T), \quad x \in \mathbb{R}^n, \\ u(0) = u_0, \quad u_t(0) = u_1, & x \in \mathbb{R}^n. \end{cases} \quad (1.1)$$

Here $u = u(t, x)$ is a real-valued unknown function, $b(t)$ is a smooth positive function, \square denotes $\partial_t^2 - \Delta_x$, and $u_0 = u_0(x), u_1 = u_1(x)$ are given initial data.

Damped wave equations are known as models describing the voltage and the current on an electrical transmission line with a resistance. It is also derived as a modified heat conduction equation from the heat balance law and the so-called Cattaneo–Vernotte law instead of the usual Fourier law (see [1, 17, 30]). The term $b(t)u_t$ is called the damping term, which prevents the motion of the wave and reduces its energy, and the coefficient $b(t)$ represents the strength of the damping.

From a mathematical point of view, it is an interesting problem to study how the damping term affects the properties of the solution. In particular, in this paper we investigate the relation between the damping term and the blow-up behavior of the solution of the Cauchy problem (1.1). To this end, as a typical case, we assume that $b(t)$ satisfies

$$b_1(1+t)^{-\beta} \leq b(t) \leq b_2(1+t)^{-\beta}, \quad |b'(t)| \leq b_3(1+t)^{-1}b(t) \quad (1.2)$$

for $t \geq 0$ with some $\beta \in \mathbb{R}$ and some positive constants b_1, b_2 and b_3 .

Since the the nonlinearity $|u|^p$ of (1.1) is a source term, in general the solution may blow up in finite time even if the initial data is sufficiently small. Indeed, for the semilinear heat equation $v_t - \Delta v = v^p$ with a nonnegative initial data $v(0) = v_0 \geq 0$, Fujita [8] found that there is the critical exponent $p_F = 1 + 2/n$, that is, if $p > p_F$, then the global solution uniquely exists for suitably small initial data comparing with the Gaussian; if $1 < p < p_F$, then all positive solutions blow

up in finite time. Later on, it is shown that the critical case $p = p_F$ belongs to the blow-up region (see Hayakawa [12] and Kobayashi, Sino and Tanaka [16]).

The blow-up of solutions of semilinear damped wave equations was firstly studied by Li and Zhou [18]. They treated the so-called classical damping case, that is, (1.1) with $b(t) \equiv 1$, and proved that when $n = 1$ or $n = 2$ and $p \leq p_F$, if the initial data satisfy $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} (u_0 + u_1)(x) dx > 0$, then the local solution blows up within a finite time. Moreover, they obtained the sharp upper bound of the lifespan in terms of the size of the initial data. Namely, denoting $u_0 = \varepsilon a_0, u_1 = \varepsilon a_1$ with $\varepsilon > 0$ and $a_0, a_1 \in C_0^\infty(\mathbb{R}^n)$ having positive average, they proved that the lifespan (maximal existence time of the local solution) T_0 satisfies

$$T_0 \leq \begin{cases} C\varepsilon^{-\frac{1}{p-1-\frac{n}{2}}} & (1 < p < p_F), \\ \exp(C\varepsilon^{-(p-1)}) & (p = p_F). \end{cases} \quad (1.3)$$

Furthermore, by their method, we can prove the estimate

$$I(t) \geq C(CI(0) - t)^{-\frac{2}{p-1}} \quad \text{with} \quad I(t) = \int_{\mathbb{R}^n} u(t, x) dx.$$

This shows the blow-up rate of the average of the solution. However, their argument depends on the positivity of the fundamental solution of the damped wave equation, which is valid only in $n \leq 3$, and it cannot be applied to higher dimensional cases or variable coefficient cases. They also proved the global existence of solutions for small initial data when $p > p_F$. Therefore, they determined the critical exponent for the classical damped wave equation for $n \leq 2$ for smooth and compactly supported initial data. Here, we say that a number $p_c > 1$ is the critical exponent for the semilinear damped wave equation (1.1) if $p > p_c$, then the global solution uniquely exists for sufficiently small data; if $p \leq p_c$, then the local solution blows up in finite time, provided that the data has certain positive average determined from $b(t)$.

Later on, for $n = 3$, Nishihara [24, 25] discovered a decomposition of the linear solution

$$S_n(t)u_1(x) = J_n(t)u_1(x) + e^{-\frac{t}{2}}W_n(t)u_1(x),$$

where $S_n(t), W_n(t)$ are the fundamental solution of the linear damped wave equation $\square u + u_t = 0$ and the linear wave equation $\square u = 0$, respectively, and $J_n(t)u_1$ behaves as the solution of the linear heat equation $v_t - \Delta v = 0$. Then, he proved the small data global existence when $p > p_F$ and the sharp upper bound of the lifespan (1.3) when $p \leq p_F$. For $n = 1, 2$ and $n \geq 4$, the same type decomposition was obtained by Marcati and Nishihara [20], Hosono and Ogawa [13] and Narazaki [23] (see also Sakata and the third author [27] for the exact decomposition for $n \geq 4$).

For higher dimensional cases $n \geq 4$, Todorova and Yordanov [29] and Zhang [35] determined the critical exponent as $p = p_F$.

Concerning the estimate of the lifespan for $n \geq 4$, for the subcritical case $p < p_F$, the second and the third author [15] showed an almost sharp estimate of the lifespan

$$C_1\varepsilon^{-\frac{1}{p-1-\frac{n}{2}}+\delta} \leq T_0 \leq C_2\varepsilon^{-\frac{1}{p-1-\frac{n}{2}}}$$

for small $\varepsilon > 0$ with arbitrary small $\delta > 0$ and some constants $C_1, C_2 > 0$. For the critical case $p = p_F$, the second author and Ogawa [14] obtained

$$\exp(C_1\varepsilon^{-(p-1)}) \leq T_0 \leq \exp(C_2\varepsilon^{-p})$$

with some constant $C_1, C_2 > 0$ (see Proposition 1.5 below). As in the case $n \leq 3$, we expect that the sharp upper estimate of the lifespan is given by $T_0 \leq \exp(C\varepsilon^{-(p-1)})$ for higher dimensional cases $n \geq 4$. However, this problem is still open.

In regard to the lifespan estimate for the semilinear wave equation with time-dependent damping (1.1), much less is known. When $b(t) = (1+t)^{-\beta}$ with $\beta \in (-1, 1)$ and $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is compactly supported, Nishihara [26] and Lin, Nishihara and Zhai [19] proved that the critical exponent is given by $p = p_F$ (see also D'Abbico, Lucente and Reissig [6] for more general damping and initial data). After that, for subcritical cases $p < p_F$, the second author and the third author [15] obtained an almost sharp estimate of the lifespan

$$C_1 \varepsilon^{-\frac{1}{(\frac{1}{p-1} - \frac{n}{2})(1+\beta)} + \delta} \leq T_0 \leq C_2 \varepsilon^{-\frac{1}{(\frac{1}{p-1} - \frac{n}{2})(1+\beta)}}$$

with arbitrary small $\delta > 0$ and some constants $C_1, C_2 > 0$.

For the case where $b(t) = b_0/(1+t)$ with $b_0 > 0$, the linearized problem of (1.1)

$$\square u + \frac{b_0}{1+t} u_t = 0$$

has scaling invariance, and it is known that the asymptotic behavior of the solution depends on the value of the constant $b_0 > 0$ (see Wirth [34]). For the semilinear problem

$$\square u + \frac{b_0}{1+t} u_t = |u|^p, \quad (1.4)$$

D'Abbico and Lucente [5] and D'Abbico [4] determined the critical exponent as $p = p_F$ when $b_0 \geq 5/3$ ($n = 1$), $b_0 \geq 3$ ($n = 2$) and $b_0 \geq n + 2$ ($n \geq 3$). Moreover, in the special case $b_0 = 2$, by setting $u = (1+t)w$, the equation (1.4) is transformed into the semilinear wave equation $\square w = (1+t)^{-(p-1)}|w|^p$. In view of this, D'Abbico, Lucente and Reissig [7] showed that the critical exponent is given by $p_2(n) = \max\{p_F, p_0(n+2)\}$ for $n \leq 3$, where $p_0(m)$ is the positive root of $(m-1)p^2 - (m+1)p - 2 = 0$. These results were recently extended to scale-invariant mass and dissipation by Nasciment, Palmieri and Reissig [22]. Wakasa [31] obtained the optimal estimate of the lifespan of solutions to (1.4) with $b_0 = 2$ for $n = 1$:

$$T_0 \sim \begin{cases} C\varepsilon^{-\frac{p-1}{3-p}} & (p < 3), \\ \exp(C\varepsilon^{-(p-1)}) & (p = 3). \end{cases}$$

However, in the general case $b_0 \neq 2$, the optimal lifespan estimate is not known, while partial results were given in [32].

When $\beta = -1$, the third author [33] recently studied the global existence and asymptotic behavior for $p > p_F$. However, there are no results about blow-up and estimates of the lifespan for $p \leq p_F$.

Finally, for $\beta < -1$, we expect that for any $p > 1$, the global solution uniquely exists for sufficiently small initial data. This problem will be discussed elsewhere. When $\beta > 1$, we expect that the critical exponent is given by $p_0(n)$, that is, the critical exponent coincides with that of the semilinear wave equation without damping. However, this is still an open problem.

In this paper, we give the sharp upper estimate of lifespan for subcritical nonlinearities $p < p_F$ and the effective damping $\beta \in [-1, 1)$. The case $\beta = -1$ is completely new. We also prove the sharp lower estimate the lifespan when $p = p_F$

and $\beta \in [-1, 1)$. For the case $\beta = 1$, some upper estimates of the lifespan will be given, while it seems not to be optimal in general.

To state our main results, we first introduce the definition of strong solutions:

Definition 1.1. Let $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and let $T \in (0, \infty]$. A function

$$u \in C^2([0, T]; H^{-1}(\mathbb{R}^n)) \cap C^1([0, T]; L^2(\mathbb{R}^n)) \cap C([0, T]; H^1(\mathbb{R}^n))$$

is called a strong solution of the Cauchy problem (1.1) on $[0, T)$ if u satisfies the initial conditions $u(0) = u_0, u_t(0) = u_1$ and the first equation of (1.1) in $C^2([0, T]; H^{-1}(\mathbb{R}^n))$.

When $1 < p < \infty$ ($n = 1, 2$), $1 < p \leq n/(n-2)$ ($n \geq 3$), for any $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, it is well-known that there exist $T > 0$ and a unique strong solution u of the Cauchy problem (1.1) on $[0, T)$ (see [28] or [21]). We will show the existence of the strong solution for some $T > 0$ in the appendix for the reader's convenience.

Definition 1.2. The lifespan T_0 of a solution u for the Cauchy problem (1.1) is defined by

$$T_0 = \sup\{T > 0 \mid u \text{ is a strong solution for (1.1) on } [0, T)\}.$$

To state our main results, we introduce assumptions and notations. We recall $p_F = 1 + 2/n$. In what follows, we assume that the coefficient of the damping term $b(t)$ is a smooth function satisfying (1.2) with some $\beta \in [-1, 1]$. We put

$$B(t) = \int_0^t b(\tau)^{-1} d\tau.$$

Then, by the assumption (1.2), $B(t)$ satisfies

$$\begin{cases} B_1(1+t)^{1+\beta} \leq B(t) \leq B_2(1+t)^{1+\beta} & (\beta \in (-1, 1]), \\ B_1 \log(2+t) \leq B(t) \leq B_2 \log(2+t) & (\beta = -1) \end{cases} \quad (1.5)$$

for $t \geq 1$ with some constants $B_1, B_2 > 0$ (the second inequalities of each case are still valid for any $t \geq 0$). Note that the function $B(t)$ is strictly increasing due to the positivity of $b(t)$, and hence, $B(t)$ has the inverse function $B^{-1}(\tau)$ satisfying

$$\begin{cases} B_3(1+\tau)^{\frac{1}{1+\beta}} \leq B^{-1}(\tau) \leq B_4(1+\tau)^{\frac{1}{1+\beta}} & (\beta \in (-1, 1]), \\ \exp(B_3(1+\tau)) \leq B^{-1}(\tau) \leq \exp(B_4(1+\tau)) & (\beta = -1) \end{cases} \quad (1.6)$$

for $\tau \geq 1$ with some constants $B_3, B_4 > 0$ (the second inequalities of each case are still valid for any $\tau \geq 0$). We also remark that the changing variable $s = B(t)$ transforms the corresponding parabolic problem $b(t)v_t - \Delta v = |v|^p$ into $\tilde{v}_s - \Delta \tilde{v} = |\tilde{v}|^p$ with $v(t, x) = \tilde{v}(B(t), x)$. Therefore, the function $B(t)$ acts as a scaling function for the time variable.

Next, we define $\tilde{\psi} \in C^\infty([0, \infty); [0, 1])$ as

$$\tilde{\psi}(r) = \begin{cases} 1 & \text{if } r \leq 1, \\ \searrow & \text{if } 1 < r < 2, \\ 0 & \text{if } r \geq 2. \end{cases}$$

Let $\psi : \mathbb{R}^n \ni x \mapsto \tilde{\psi}(|x|)$ and let $\psi_R(x) = \psi(x/R)$. For $p > 1$, $\beta \in [-1, 1]$ and $A > 0$, we define $\mu(p, b, \beta, A)$ by

$$\mu(p, b, \beta, A) = \min \left\{ 1, \frac{p-1}{2} b(0) A, \left[\frac{2(p+1)}{(p-1)^2} b_1^{-2} \left\{ 2^{\frac{1}{1+\beta}} (1+B_4) \right\}^{\max(0, 2\beta)} + \frac{2(b_1^{-1} b_3 + 1)}{p-1} \right]^{-1} \right\}.$$

Here, we interpret the term $\{2^{\frac{1}{1+\beta}} (1+B_4)\}^{\max(0, 2\beta)} = 1$ if $\beta = -1$. We also note that $A_1 \leq A_2$ implies $\mu(p, b, \beta, A_1) \leq \mu(p, b, \beta, A_2)$. The constant $\mu(p, b, \beta, A)$ appears as the coefficient of the initial data in the estimate of the lifespan and the lower estimate of the average of the solution (see Proposition 1.3). For $n \in \mathbb{N}$, $p > 1$, $\ell \in \mathbb{N}$ satisfying $\ell > 2p'$, and $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi \geq 0$, we also define $A(n, p, \ell, \phi)$ as

$$A(n, p, \ell, \phi) = 2^{p'-1} p'^{-\frac{1}{p}} p^{\frac{1-p'}{p}} \|\Phi^{p'} \phi^{\ell-2p'}\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}} \|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p'}},$$

where $p' = p/(p-1)$ and

$$\Phi = \phi^{2-\ell} \Delta(\phi^\ell) = \ell(\ell-1) \nabla \phi \cdot \nabla \phi + \ell \phi \Delta \phi.$$

We will derive an ordinary differential inequality for the weighted average of the solution up to the constant $A(n, p, \ell, \phi)$.

Now we are in a position to give our main results. The first one is the upper estimate of the lifespan of solutions to (1.1) in a general setting. At the moment we do not need any condition on p such as $p \leq p_F$ but we impose certain condition with respect to the test function ϕ .

Proposition 1.3. *Let $\beta \in [-1, 1]$, $p \in (1, \infty)$ and $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, and let u be the associated strong solution on $[0, T_0)$ with the lifespan T_0 . Assume that there exists $\phi \in \mathcal{S}(\mathbb{R}^n; [0, \infty))$ such that*

$$0 < I_\phi(0) - A(n, p, \ell, \phi) < 2^{\frac{1}{p-1}} \|\phi^\ell\|_{L^1(\mathbb{R}^n)}, \quad I'_\phi(0) > 0, \quad (1.7)$$

where

$$I_\phi(t) = \int_{\mathbb{R}^n} u(t, x) \phi^\ell(x) dx.$$

Let

$$\begin{aligned} J_\phi(t) &= I_\phi(t) - A(n, p, \ell, \phi), \\ \tilde{J}_\phi(0) &= 2^{-\frac{1}{p-1}} \|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{-1} J_\phi(0), \\ A_1 &= \frac{J'_\phi(0)}{J_\phi(0)} = \frac{I'_\phi(0)}{I_\phi(0) - A(n, p, \ell, \phi)}. \end{aligned}$$

Then, we have

$$J_\phi(t) \geq J_\phi(0) \left(1 - \mu(p, b, \beta, A_1) \tilde{J}_\phi(0)^{p-1} B(t) \right)^{-\frac{2}{p-1}} \quad (1.8)$$

for $t \in [0, T_0)$. Moreover, the lifespan T_0 of the solution u is estimated as

$$T_0 \leq B^{-1} \left(\mu(p, b, \beta, A_1)^{-1} \tilde{J}_\phi(0)^{1-p} \right).$$

Proposition 1.3 implies that (1.7) is a sufficient condition for the blow-up of the solution. The condition (1.7) is related with the scaling of the equation. Indeed, letting $p \in (1, p_F)$ and taking the test function $\phi = \psi_{R(\varepsilon)}$ with an appropriate scaling parameter $R(\varepsilon)$, we ensure the condition (1.7) and show the sharp estimate of the lifespan of solutions to (1.1).

Corollary 1.4. *Let $\beta \in [-1, 1]$, $p \in (1, p_F)$ and $(u_0, u_1) = \varepsilon(a_0, a_1)$ with $\varepsilon > 0$. We assume that $(a_0, a_1) \in (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$ satisfy*

$$I_0 = \int_{|x| < R} a_0(x) dx > 0, \quad I_1 = \int_{|x| < R} a_1(x) dx > 0.$$

Let

$$R(\varepsilon) = A(n, p, \ell, \psi)^{\frac{p-1}{n(p_F-p)}} \left(\frac{\varepsilon}{4} I_0 \right)^{-\frac{p-1}{n(p_F-p)}} \quad (1.9)$$

and let $\varepsilon_0 > 0$ satisfy that

$$\int_{\mathbb{R}^n} \psi_{R(\varepsilon_0)}^\ell(x) a_0(x) dx \geq \frac{1}{2} I_0, \quad (1.10)$$

$$\int_{\mathbb{R}^n} \psi_{R(\varepsilon_0)}^\ell(x) a_1(x) dx \geq \frac{1}{2} I_1, \quad (1.11)$$

$$\varepsilon_0 I_0 \leq 2^{\frac{1}{p-1}} \|\psi^\ell\|_{L^1(\mathbb{R}^n)} R(\varepsilon_0)^n. \quad (1.12)$$

Then, for any $\varepsilon \in (0, \varepsilon_0]$, the associated strong solution u of (1.1) satisfies,

$$\int_{\mathbb{R}^n} u(t, x) \psi_{R(\varepsilon)}^\ell(x) dx \geq \frac{\varepsilon}{4} I_0 \left(1 - \mu_0 \varepsilon^{\frac{1}{p-1} - \frac{n}{2}} B(t) \right)^{-\frac{2}{p-1}} \quad (1.13)$$

with some constant $\mu_0 = \mu_0(n, p, b, \beta, \ell, \psi, I_0, I_1) > 0$ and the lifespan $T_0 = T_0(\varepsilon)$ is estimated as

$$T_0 \leq B^{-1} \left(\mu_0^{-1} \varepsilon^{-\frac{1}{p-1} - \frac{n}{2}} \right). \quad (1.14)$$

Remark 1.1. The constant μ_0 is given by (3.3). Also, under the assumptions in Corollary 1.4, combining (1.14) and (1.6), we see that

$$T_0 \leq \begin{cases} B_4 \left(1 + \mu_0^{-1} \varepsilon^{-\frac{1}{p-1} - \frac{n}{2}} \right)^{\frac{1}{1+\beta}} & (\beta \in (-1, 1]), \\ \exp \left(B_4 \left(1 + \mu_0^{-1} \varepsilon^{-\frac{1}{p-1} - \frac{n}{2}} \right) \right) & (\beta = -1). \end{cases}$$

Propositions 1.3 and 1.4 show the blow-up behavior for the solutions of (1.1) and how it depends on the parameter β . They are summarized in the following way:

- The blow-up rate of the solution near the blow-up time is similar to that of the nonlinear wave equation, though the time variable is scaled by $B(t)$.
- On the other hand, the estimate of the lifespan of the solution is similar to that of the nonlinear heat equation.

Concerning the upper estimate of the lifespan T_0 in the critical case $p = p_F$, we refer the reader to a recent result of the second author and Ogawa [14, Theorem 2.5]:

Proposition 1.5 ([14]). *Let $b(t) = (1+t)^{-\beta}$, $\beta \in (-1, 1)$, $p = p_F$ and let $(u_0, u_1) = \varepsilon(a_0, a_1)$ with $\varepsilon > 0$, and we assume that $(a_0, a_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and (a_0, a_1) satisfies*

$$B_0 a_0 + a_1 \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} (B_0 a_0 + a_1)(x) dx > 0,$$

where

$$B_0 = \left(\int_0^\infty \exp \left(- \int_0^t (1+s)^{-\beta} ds \right) dt \right)^{-1}.$$

Then, there exists a constant $C = C(n, \beta, a_0, a_1) > 0$ such that the lifespan $T_0 = T_0(\varepsilon)$ of the associated strong solution is estimated as

$$T_0 \leq \exp(C\varepsilon^{-p})$$

for any $\varepsilon \in (0, 1]$.

Next, we discuss the optimality of the estimate (1.14) with respect to the power of ε , that is, the estimate of the lifespan from below. Following the third author's recent work [33], we have the lower estimate of the lifespan.

Proposition 1.6. *Let $\beta \in [-1, 1)$, $p \in (1, p_F)$ and let $(u_0, u_1) = \varepsilon(a_0, a_1)$ with $\varepsilon > 0$. We assume that $(a_0, a_1) \in H^{1,m}(\mathbb{R}^n) \times H^{0,m}(\mathbb{R}^n)$ with $m = 1$ ($n = 1$), $m > n/2 + 1$ ($n \geq 2$). Then, there exist constants $\varepsilon_1 = \varepsilon(n, \beta, p, m, \|a_0\|_{H^{1,m}}, \|a_1\|_{H^{0,m}}) > 0$ and $C_* = C_*(n, \beta, p, m, \|a_0\|_{H^{1,m}}, \|a_1\|_{H^{0,m}}) > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, the lifespan $T_0 = T_0(\varepsilon)$ of the associated strong solution is estimated by*

$$T_0 \geq B^{-1} \left(C_* \varepsilon^{-\frac{1}{p-1-\frac{n}{2}}} \right).$$

Remark 1.2. *Under the assumptions in Proposition 1.6, combining the above estimate and (1.6), we see that*

$$T_0 \geq \begin{cases} C_* \varepsilon^{-\frac{1}{(\frac{1}{p-1}-\frac{n}{2})(1+\beta)}} & (\beta \in (-1, 1)), \\ \exp \left(C_* \varepsilon^{-\frac{1}{p-1-\frac{n}{2}}} \right) & (\beta = -1). \end{cases}$$

In the case $\beta \in (-1, 1)$, the rate of ε coincides with that of Corollary 1.4. Namely, we have the sharp estimate of the lifespan of the solution. In the case $\beta = -1$, we have an exponential lower bound, which is the so-called almost global existence of the solution. This is quite reasonable because in this case where the damping is very strong, it helps well the solution exist longer time.

On the other hand, in the critical case $p = p_F$, we have the following:

Proposition 1.7. *Let $\beta \in [-1, 1)$, $p = p_F$ and let $(u_0, u_1) = \varepsilon(a_0, a_1)$ with $\varepsilon > 0$. We assume that $(a_0, a_1) \in H^{1,m}(\mathbb{R}^n) \times H^{0,m}(\mathbb{R}^n)$ with $m = 1$ ($n = 1$), $m > n/2 + 1$ ($n \geq 2$). Then, there exist constants $\varepsilon_2 = \varepsilon(n, \beta, p, m, \|a_0\|_{H^{1,m}}, \|a_1\|_{H^{0,m}}) > 0$ and $C_* = C_*(n, \beta, p, m, \|a_0\|_{H^{1,m}}, \|a_1\|_{H^{0,m}}) > 0$ such that for any $\varepsilon \in (0, \varepsilon_2]$, the lifespan $T_0 = T_0(\varepsilon)$ of the associated strong solution is estimated by*

$$T_0 \geq B^{-1} \left(\exp \left(C_* \varepsilon^{-(p-1)} \right) \right).$$

Remark 1.3. *Under the assumptions in Proposition 1.7, combining the above estimate and (1.6), we see that*

$$T_0 \geq \begin{cases} \exp\left(C_* \varepsilon^{-(p-1)}\right) & (\beta \in (-1, 1)), \\ \exp\left(\exp\left(C_* \varepsilon^{-(p-1)}\right)\right) & (\beta = -1). \end{cases}$$

Proposition 1.7 shows that in the critical case, we have the exponential and the double-exponential estimate from the below for the case $\beta \in (-1, 1)$ and $\beta = -1$, respectively. Comparing with Proposition 1.6, it is also quite reasonable. We also remark that Propositions 1.6 and Proposition 1.7 are still true if we replace the nonlinearity $|u|^p$ by $\pm|u|^{p-1}u$ or $-|u|^p$.

Our results with the previous ones are summarized in Table 1, where we consider the damping $b_0(1+t)^{-\beta}$ with $b_0 > 0$ and $\beta \in [-1, 1]$.

$\beta \setminus p$	$1 < p < p_F$	$p = p_F$
$\beta = -1$	$T_0 \sim \exp\left(C\varepsilon^{-\frac{1}{\frac{1}{p-1}-\frac{n}{2}}}\right)$	$\exp\left(\exp\left(C\varepsilon^{-(p-1)}\right)\right) \leq T_0$
$-1 < \beta < 1$	$T_0 \sim C\varepsilon^{-\frac{1}{(\frac{1}{p-1}-\frac{n}{2})(1+\beta)}}$	$\exp\left(C\varepsilon^{-(p-1)}\right) \leq T_0 \leq \exp\left(C\varepsilon^{-p}\right)$ (upper bound is by [14])
$\beta = 1$	$T_0 \leq C\varepsilon^{-\frac{1}{2(\frac{1}{p-1}-\frac{n}{2})}},$ $T_0 \sim \varepsilon^{-\frac{p-1}{3-p}}$ for $n = 1, b_0 = 2$ ([31])	open (in general), $T_0 \sim \exp\left(C\varepsilon^{-(p-1)}\right)$ for $n = 1, b_0 = 2$ ([31])

TABLE 1. Estimates of lifespan

When $1 < p < p_F$ and $\beta = 1$, we have an upper bound of T_0 , while it seems not to be optimal in general. In this case it is known that the critical exponent may change (see D’Abicco, Lucente and Reissig [7] and Wakasa [31]).

We mention about the strategy of the proof. Our method is a hybrid version of the method by Li and Zhou [18] and by Zhang [35]. The method by Li and Zhou [18] is based on a ordinary differential inequality. However, in order to derive an ordinary differential inequality from the damped wave equation, their argument requires the positivity of the fundamental solution, which fails in higher dimensional cases. The method by Zhang [35] is the so-called test function method. He considered an average of the nonlinearity of the solution $\int_0^\infty \int_{\mathbb{R}^n} |u(t, x)|^p \psi_R(t, x) dx dt$ with suitable family of cut-off functions $\{\psi_R\}_{R>0}$, and leads to contradiction by the integration by parts and a scaling argument. However, this method is based on contradiction argument, and the mechanism of the blow-up is unclear. Moreover, by this approach, we cannot obtain blow-up rates of solutions.

To overcome these difficulties, we employ the method developed by the first author and Ozawa [9, 10] in which the lifespan of the solution for a nonlinear Schrödinger equation is studied. They considered a localized average of the solution $\int_{\mathbb{R}^n} u(t, x) \phi(x) dx$ with a cut-off function $\phi(x)$, and derive an ordinary differential inequality for it. Then, they showed the estimate of the lifespan of the solution

of the ordinary differential inequality. In this paper, we adapt their method to the damped wave equations. First, we establish the blow-up and estimate of the lifespan of the solution to the ordinary differential inequality

$$\begin{cases} f''(t) + b(t)f'(t) \geq \gamma f(t)^p, \\ f(0) \geq \varepsilon_0, \\ f'(0) \geq A_0 \varepsilon_0, \end{cases} \quad (1.15)$$

with $A_0, \gamma, \varepsilon_0 > 0$. We remark that Li and Zhou also obtained the finite time blow-up for (1.15) and the life-span estimate. However, as far as the authors know, explicit subsolutions for (1.15) had not been known even though they are well known for a first order ordinary differential inequality $f' \geq f^p$. We construct explicit subsolutions by a comparison lemma given by Li and Zhou [18] and the blow-up rate (1.8) follows from these explicit subsolution. For detail, see Proposition 2.3. Next, to prove Proposition 1.3, we follow [9, 10] and consider the localized average $I_\phi(t) = \int_{\mathbb{R}^n} u(t, x)\phi(x)dx$, and derive the ordinary differential inequality (1.15) from the equation (1.1). Finally, for Corollary 1.4, we choose a special family of cut-off functions and apply a scaling argument to reduce its proof to Proposition 1.3.

For Propositions 1.6 and 1.7, we employ the method of scaling variables, which was originally introduced by Gallay and Raugel [11]. Coulaud [3] refined it and applied to the second grade fluids equations in three space dimensions. Recently, the third author [33] applied the method to obtain the asymptotic profile for the semilinear wave equation with time-dependent damping.

This paper is organized as follows. In section 2, we study the blow-up of solutions to the ordinary differential inequality (1.15). In section 3, applying the theory of ordinary differential inequalities prepared in Section 2, we give a proof of Proposition 1.3 and Corollary 1.4. Section 4 is devoted to the proof of Propositions 1.6 and 1.7. Finally, in the appendix, we give a proof of local existence of solutions in the energy space.

We finish this section with notations used throughout this paper. We denote by C a generic constant, which may change from line to line. In particular, we sometimes use the symbol $C(*, \dots, *)$ for constants depending on the quantities appearing in parenthesis.

We give the notations of function spaces. Let $L^p(\mathbb{R}^n)$ be the usual Lebesgue space equipped with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} \quad (1 \leq p < \infty), \quad \|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|.$$

For $s \in \mathbb{Z}_{\geq 0}, m \geq 0$, we define the weighted Sobolev space $H^{s,m}(\mathbb{R}^n)$ by

$$H^{s,m}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n); \|f\|_{H^{s,m}} < \infty\},$$

$$\|f\|_{H^{s,m}} = \left(\sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} (1 + |x|^2)^m |\partial_x^\alpha f(x)|^2 dx \right)^{1/2}.$$

In particular, when $m = 0$, we also denote $H^{s,0}(\mathbb{R}^n)$ as $H^s(\mathbb{R}^n)$. For an interval I and a Banach space X , we define $C^r(I; X)$ as the space of r -times continuously differentiable mapping from I to X with respect to the topology in X .

2. ESTIMATES OF THE LIFESPAN OF SOLUTIONS TO ORDINARY DIFFERENTIAL INEQUALITIES

In this section, we study the estimates of lifespan of solutions to the ordinary differential inequality (1.15). To this end, the following comparison theorem plays a critical role:

Lemma 2.1. [18, Lemma 3.1] *Let $T > 0$. We assume that functions $k, h \in C^2([0, T])$ satisfy*

$$\begin{cases} a(t)k''(t) + k'(t) \geq c(t)k(t)^p, \\ a(t)h''(t) + h'(t) \leq c(t)h(t)^p \end{cases}$$

for $t \in [0, T]$, where $p \geq 1$ and $a(t), c(t)$ are nonnegative continuous function on $[0, T]$. We further assume that

$$\begin{cases} k(0) > h(0), \\ k'(0) \geq h'(0). \end{cases}$$

Then, we have $k'(t) > h'(t)$ for any $t \in [0, T]$.

Thanks to Lemma 2.1, we analyze the behavior of solutions for (1.15) by comparing with subsolutions. In the next lemma, we introduce our subsolution.

Lemma 2.2. *Let $A_0 > 0$, $\beta \in [-1, 1]$, $p > 1$ and let $\varepsilon_0 \in (0, 1]$. We put*

$$T_1 = B^{-1} \left(\mu(p, b, \beta, A_0)^{-1} \varepsilon_0^{1-p} \right). \quad (2.1)$$

Moreover, for $t \in [0, T_1)$, we define

$$g(t) = \varepsilon_0 \left(1 - \mu(p, b, \beta, A_0) \varepsilon_0^{p-1} B(t) \right)^{-\frac{2}{p-1}}.$$

Then g satisfies that

$$\begin{cases} g''(t) + b(t)g'(t) \leq g(t)^p, & \text{for } t \in [0, T_1), \\ g(0) = \varepsilon_0, \\ g'(0) \leq A_0 \varepsilon_0. \end{cases}$$

Proof. For simplicity, we denote $\mu(p, b, \beta, A_0)$ as μ . Since $\mu \leq \frac{p-1}{2}b(0)A_0$, by a direct calculation, we have

$$\begin{aligned} g'(t) &= \frac{2\mu}{p-1} \varepsilon_0^p \left(1 - \mu \varepsilon_0^{p-1} B(t) \right)^{-\frac{p+1}{p-1}} b(t)^{-1}, \\ g'(0) &= \frac{2\mu}{p-1} b(0)^{-1} \varepsilon_0^p \leq A_0 \varepsilon_0, \\ g''(t) &= -\frac{2\mu}{p-1} \varepsilon_0^p \left(1 - \mu \varepsilon_0^{p-1} B(t) \right)^{-\frac{p+1}{p-1}} b'(t) b(t)^{-2} \\ &\quad + \frac{2(p+1)}{(p-1)^2} \mu^2 \varepsilon_0^{2p-1} \left(1 - \mu \varepsilon_0^{p-1} B(t) \right)^{-\frac{2p}{p-1}} b(t)^{-2}. \end{aligned}$$

Then, for $t < T_1$, we obtain

$$\begin{aligned}
& g''(t) + b(t)g'(t) \\
& \leq g(t)^p \left(\frac{2(p+1)}{(p-1)^2} \varepsilon_0^{p-1} b(t)^{-2} \mu^2 + \frac{2}{p-1} b'(t) b(t)^{-2} \mu + \frac{2}{p-1} \mu \right) \\
& \leq g(t)^p \left(\frac{2(p+1)}{(p-1)^2} 2^{\frac{2\beta}{1+\beta}} b_1^{-2} (1+B_4)^{\max(0,2\beta)} + \frac{2(b_1^{-1}b_3+1)}{p-1} \right) \mu \\
& \leq g(t)^p.
\end{aligned}$$

Here, for the second inequality, when $\beta \in (0, 1]$, we have used that

$$\begin{aligned}
\varepsilon_0^{p-1} b(t)^{-2} \mu & \leq \varepsilon_0^{p-1} \mu b_1^{-2} (1+T_1)^{2\beta} \\
& \leq \varepsilon_0^{p-1} \mu b_1^{-2} \left[1 + B^{-1} \left(\mu^{-1} \varepsilon_0^{1-p} \right) \right]^{2\beta} \\
& \leq \varepsilon_0^{p-1} \mu b_1^{-2} \left[1 + B_4 \left(1 + \mu^{-1} \varepsilon_0^{1-p} \right)^{\frac{1}{1+\beta}} \right]^{2\beta} \\
& \leq \varepsilon_0^{p-1} \mu b_1^{-2} (1+B_4)^{2\beta} \left(1 + \mu^{-1} \varepsilon_0^{1-p} \right)^{\frac{2\beta}{1+\beta}} \\
& \leq 2^{\frac{2\beta}{1+\beta}} \left(\varepsilon_0^{p-1} \mu \right)^{\frac{1-\beta}{1+\beta}} b_1^{-2} (1+B_4)^{2\beta} \\
& \leq 2^{\frac{2\beta}{1+\beta}} b_1^{-2} (1+B_4)^{2\beta},
\end{aligned}$$

and for the third inequality we have used the definition of $\mu(p, b, \beta, A_0)$. \square

Proposition 2.3. *Let $T_0 > 0$, $A_0 > 0$, $\beta \in [-1, 1]$, $p > 1$, $\gamma > 0$, and let $\varepsilon_0 \in (0, \gamma^{-\frac{1}{p-1}}]$. Assume that $f \in C^2([0, T_0])$ satisfies $f(t) > 0$ for $t \in [0, T_0]$ and*

$$\begin{cases} f''(t) + b(t)f'(t) \geq \gamma f(t)^p & \text{for } t \in [0, T_0], \\ f(0) > \varepsilon_0, \\ f'(0) \geq A_0 \varepsilon_0. \end{cases}$$

Then, with $\delta_0 = \gamma^{\frac{1}{p-1}} \varepsilon_0$, we have

$$f(t) \geq \varepsilon_0 \left(1 - \mu(p, b, \beta, A_0) \delta_0^{p-1} B(t) \right)^{-\frac{2}{p-1}}$$

for $t \in [0, T_0)$, and T_0 is estimated as

$$T_0 \leq B^{-1} \left(\mu(p, b, \beta, A_0)^{-1} \delta_0^{1-p} \right). \quad (2.2)$$

Proof. Let $\tilde{f} = \gamma^{\frac{1}{p-1}} f$ and $\delta_0 = \gamma^{\frac{1}{p-1}} \varepsilon_0$. Then, \tilde{f} satisfies

$$\begin{cases} \tilde{f}''(t) + b(t)\tilde{f}'(t) \geq \tilde{f}(t)^p & \text{for } t \in [0, T_0], \\ \tilde{f}(0) > \delta_0, \\ \tilde{f}'(0) \geq A_0 \delta_0. \end{cases}$$

Let T_1 be defined in (2.1) with $\varepsilon_0 = \delta_0$, that is, T_1 is the right-hand side of (2.2). For $\rho \in [0, 1)$, we put $\delta_\rho = (1 - \rho)\delta_0$ and define

$$\tilde{g}_\rho(t) = \delta_\rho \left(1 - \mu(p, b, \beta, A_0) \delta_\rho^{p-1} B(t) \right)^{-\frac{2}{p-1}}$$

for $t \in [0, T_1]$. Noting $\delta_0 \in (0, \mu(p, b, \beta, A_0)^{-\frac{1}{p-1}}]$ and applying Lemma 2.2, we see that \tilde{g}_ρ satisfies

$$\begin{cases} \tilde{g}_\rho''(t) + b(t)\tilde{g}_\rho'(t) \leq \tilde{g}_\rho(t)^p & \text{for } t \in [0, T_1], \\ \tilde{g}_\rho(0) = \delta_\rho, \\ \tilde{g}_\rho'(0) \leq A_0\delta_\rho. \end{cases}$$

We put $T_2 = \min(T_0, T_1)$. Then, by Lemma 2.1, for any $\rho \in (0, 1)$, we have $\tilde{f}(t) \geq \tilde{g}_\rho(t)$ for $t \in [0, T_2]$. Noting the continuity of \tilde{g}_ρ with respect to $\rho \in [0, 1]$ and letting $\rho \rightarrow 0$, we see that $\tilde{f}(t) \geq \tilde{g}_0(t)$ holds for any $t \in [0, T_2]$.

Next, we see that $T_2 = T_0$. Indeed, if $T_0 > T_1$, namely $T_2 = T_1$, then $\tilde{f}(t)$ is defined as a C^2 function on the interval $[0, T_1]$. However, by the definition of \tilde{g}_0 , we immediately obtain $\lim_{t \rightarrow T_1-0} \tilde{g}_0(t) = \infty$. This and the fact $\tilde{f}(t) \geq \tilde{g}_0(t)$ for $t \in [0, T_1]$ imply $\lim_{t \rightarrow T_1-0} \tilde{f}(t) = \infty$, which contradicts $\tilde{f} \in C^2([0, T_1])$. Consequently, we have $T_2 = T_0$, namely $T_0 \leq T_1$ and we complete the proof. \square

3. PROOF OF THE PROPOSITION 1.3 AND COROLLARY 1.4

Proof of Proposition 1.3. Let u be a strong solution of (1.1) on $[0, T_0]$ with the lifespan T_0 . Let $\phi \in \mathcal{S}(\mathbb{R}^n; [0, \infty))$ satisfy the inequality (1.7). Recall that $I_\phi(t) = \int_{\mathbb{R}^n} u(t, x)\phi^\ell(x)dx$ and $\Phi(x) = \ell(\ell-1)\nabla\phi(x) \cdot \nabla\phi(x) + \ell\phi(x)\Delta\phi(x)$. Then, by the continuity of $I_\phi(t)$ with respect to t , there exists $t_0 > 0$ such that $I_\phi(t) - A(n, p, \ell, \phi) > 0$ holds for $t \in [0, t_0]$. By a direct calculation, we have for $t \in [0, t_0]$,

$$\begin{aligned} \frac{d^2}{dt^2}I_\phi(t) + b(t)\frac{d}{dt}I_\phi(t) &= \int_{\mathbb{R}^n} (\partial_t^2 + b(t)\partial_t)u(t, x)\phi^\ell(x)dx \\ &= \int_{\mathbb{R}^n} u(t, x)\Delta(\phi^\ell(x))dx + \|u(t)\phi^{\frac{\ell}{p}}\|_{L^p(\mathbb{R}^n)}^p \\ &= \int_{\mathbb{R}^n} u(t, x)\Phi(x)\phi^{\ell-2}(x)dx + \|u(t)\phi^{\frac{\ell}{p}}\|_{L^p(\mathbb{R}^n)}^p \\ &\geq -\|\Phi\phi^{\frac{\ell}{p'}-2}\|_{L^{p'}(\mathbb{R}^n)}\|u(t)\phi^{\frac{\ell}{p}}\|_{L^p(\mathbb{R}^n)} + \|u(t)\phi^{\frac{\ell}{p}}\|_{L^p(\mathbb{R}^n)}^p \\ &\geq -2^{\frac{p'}{p}}p'^{-1}p^{-\frac{p'}{p}}\|\Phi\phi^{\frac{\ell}{p'}-2}\|_{L^{p'}(\mathbb{R}^n)}^{p'} + 2^{-1}\|u(t)\phi^{\frac{\ell}{p}}\|_{L^p(\mathbb{R}^n)}^p \\ &\geq -2^{p'-1}p'^{-1}p^{1-p'}\|\Phi^{p'}\phi^{\ell-2p'}\|_{L^1(\mathbb{R}^n)} + 2^{-1}\|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{1-p}I_\phi(t)^p \\ &= 2^{-1}\|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{1-p}(I_\phi(t)^p - A(n, p, \ell, \phi)^p) \\ &\geq 2^{-1}\|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{1-p}(I_\phi(t) - A(n, p, \ell, \phi))^p. \end{aligned}$$

Here we note that the above inequality holds as long as $I_\phi(t) - A(n, p, \ell, \phi) > 0$. The above inequality implies that $J_\phi(t) = I_\phi(t) - A(n, p, \ell, \phi)$ satisfies

$$\begin{cases} J_\phi''(t) + b(t)J_\phi'(t) \geq 2^{-1}\|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{1-p}J_\phi(t)^p & \text{for } t \in [0, t_0], \\ J_\phi(0) = I_\phi(0) - A(n, p, \ell, \phi), \\ J_\phi'(0) = A_1J_\phi(0), \end{cases} \quad (3.1)$$

where $A_1 = I_\phi'(0)/(I_\phi(0) - A(n, p, \ell, \phi))$, which is a positive constant thanks to the assumption (1.7). Moreover, by the assumption (1.7), we have $I_\phi(0) - A(n, p, \ell, \phi) \leq 2^{\frac{1}{p-1}}\|\phi^\ell\|_{L^1(\mathbb{R}^n)}$. Thus, we apply Proposition 2.3 with $\varepsilon_0 = I_\phi(0) - A(n, p, \ell, \phi)$,

$\gamma = 2^{-1} \|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{1-p}$ and $f(t) = J_\phi(t)$ to obtain

$$J_\phi(t) \geq J_\phi(0) \left(1 - \mu(p, b, \beta, A_1) \tilde{J}_\phi(0)^{p-1} B(t)\right)^{-\frac{2}{p-1}} \quad (3.2)$$

for $t \in [0, t_0)$, where $\tilde{J}_\phi(0) = 2^{-\frac{1}{p-1}} \|\phi^\ell\|_{L^1(\mathbb{R}^n)}^{-1} J_\phi(0)$.

Next, we show that $J_\phi(t) > 0$ holds for any $t \in [0, T_0)$. Indeed, if $J_\phi(t_*) = 0$ holds for some $t_* \in (0, T_0)$ and $J_\phi(t) > 0$ holds for $t \in [0, t_*)$, then, applying the same argument above, we can prove the estimate (3.2) for $t \in [0, t_*)$. However, the right-hand side of (3.2) remains positive for $t = t_*$, which contradicts $J_\phi(t_*) = 0$. Thus, we have $J_\phi(t) > 0$ for any $t \in [0, T_0)$, and $J_\phi(t)$ also satisfies the estimate (3.2) for $t \in [0, T_0)$. Hence, Proposition 2.3 with $\delta_0 = \tilde{J}_\phi(0)$ gives the desired estimates for $J_\phi(t)$ and T_0 . \square

Proof of Corollary 1.4. Let $R(\varepsilon)$ be given by (4.4). Since $n - 2\frac{p'}{p} = \frac{n(p-p_F)}{p-1}$, we calculate

$$A(n, p, \ell, \psi_{R(\varepsilon)}) = A(n, p, \ell, \psi) R(\varepsilon)^{\frac{n(p-p_F)}{p-1}} = \frac{\varepsilon}{4} I_0.$$

From this and the assumption (1.10), we obtain

$$I_{\psi_{R(\varepsilon)}} - A(n, p, \ell, \psi_{R(\varepsilon)}) \geq \frac{\varepsilon}{4} I_0.$$

Also, the assumption (1.11) immediately implies $I'_{\psi_{R(\varepsilon)}} \geq \frac{\varepsilon}{2} I_1$. Finally, the assumption (1.12) leads to

$$\begin{aligned} I_{\psi_{R(\varepsilon)}} - A(n, p, \ell, \psi_{R(\varepsilon)}) &\leq \varepsilon I_0 \\ &\leq 2^{\frac{1}{p-1}} \|\psi^\ell\|_{L^1(\mathbb{R}^n)} R(\varepsilon)^n \\ &= 2^{\frac{1}{p-1}} \|\psi_{R(\varepsilon)}^\ell\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Therefore, the condition (1.7) is fulfilled and Proposition 1.3 with $\phi = \psi_{R(\varepsilon)}$ implies that

$$J_{\psi_{R(\varepsilon)}}(t) \geq J_{\psi_{R(\varepsilon)}}(0) \left(1 - \mu(p, b, \beta, A_1(\varepsilon)) \tilde{J}_{\psi_{R(\varepsilon)}}(0)^{p-1} B(t)\right)$$

and the lifespan T_0 is estimated as

$$T_0 \leq B^{-1} \left(\mu(p, b, \beta, A_1(\varepsilon))^{-1} \tilde{J}_{\psi_{R(\varepsilon)}}(0)^{1-p} \right),$$

where

$$A_1(\varepsilon) = \frac{I'_{\psi_{R(\varepsilon)}}(0)}{I'_{\psi_{R(\varepsilon)}}(0) - A(n, p, \ell, \psi_{R(\varepsilon)})}.$$

Now, we again use the assumptions (1.10) and (1.11) to obtain

$$A_1(\varepsilon) \geq \frac{\frac{\varepsilon}{2} I_1}{\varepsilon I_0 - \frac{\varepsilon}{4} I_0} = \frac{2I_1}{3I_0}.$$

Moreover, we calculate

$$\begin{aligned} \tilde{J}_{\psi_{R(\varepsilon)}}(0)^{p-1} &= 2^{-1} \|\psi_{R(\varepsilon)}^\ell\|_{L^1(\mathbb{R}^n)}^{-(p-1)} J_{\psi_{R(\varepsilon)}}(0)^{p-1} \\ &\geq 2^{-1} \|\psi^\ell\|_{L^1(\mathbb{R}^n)}^{-(p-1)} R(\varepsilon)^{-n(p-1)} \left(\frac{\varepsilon}{4} I_0\right)^{p-1} \\ &= 2^{-1} \|\psi^\ell\|_{L^1(\mathbb{R}^n)}^{-(p-1)} A(n, p, \ell, \psi)^{-\frac{(p-1)^2}{p_F-p}} \left(\frac{\varepsilon}{4} I_0\right)^{\frac{1}{p-1} - \frac{n}{2}}. \end{aligned}$$

Consequently, letting

$$\begin{aligned} & \mu_0(n, p, b, \beta, \ell, \psi, I_0, I_1) \\ &= \mu\left(p, b, \beta, \frac{2I_1}{3I_0}\right) 2^{-1} \|\psi^\ell\|_{L^1(\mathbb{R}^n)}^{-(p-1)} A(n, p, \ell, \psi)^{-\frac{(p-1)^2}{pF-p}} \left(\frac{1}{4}I_0\right)^{\frac{1}{p-1}-\frac{p}{2}}, \end{aligned} \quad (3.3)$$

we have the desired estimates (1.13) and (1.14). \square

4. PROOFS OF PROPOSITIONS 1.6 AND 1.7

4.1. Scaling variables, local existence and spectral decomposition.

We give proofs of Propositions 1.6 and 1.7. Sections 4.1–4.3 are almost the same as in [33] and we present only their outlines. Following [33], we introduce the scaling variables

$$s = \log(B(t) + 1), \quad y = (B(t) + 1)^{-1/2}x. \quad (4.1)$$

Also, we use the notation $t(s) = B^{-1}(e^s - 1)$. We change the coordinate and the unknown function as

$$\begin{aligned} u(t, x) &= (B(t) + 1)^{-n/2} v(\log(B(t) + 1), (B(t) + 1)^{-1/2}x), \\ u_t(t, x) &= b(t)^{-1} (B(t) + 1)^{-n/2-1} w(\log(B(t) + 1), (B(t) + 1)^{-1/2}x). \end{aligned} \quad (4.2)$$

Then, the equation (1.1) is transformed into the first order system

$$\begin{cases} v_s - \frac{y}{2} \cdot \nabla_y v - \frac{n}{2} v = w, & s > 0, y \in \mathbb{R}^n, \\ \frac{e^{-s}}{b(t(s))^2} \left(w_s - \frac{y}{2} \cdot \nabla_y w - \left(\frac{n}{2} + 1 \right) w \right) + w = \Delta_y v + r(s, y), & s > 0, y \in \mathbb{R}^n, \\ v(0, y) = v_0(y) = \varepsilon a_0(y), \quad w(0, y) = w_0(y) = \varepsilon a_1(y), & y \in \mathbb{R}^n, \end{cases} \quad (4.3)$$

where

$$r(s, y) = \frac{b'(t(s))}{b(t(s))^2} w + e^{\frac{n}{2}(pF-p)s} |v|^p. \quad (4.4)$$

The local well-posedness for the system (4.3) was obtained by [33, Proposition 3.6]. In this paper, the solution satisfying certain integral equation is constructed (mild solution). Such solution also satisfies the condition of our strong solution (see Definition 1.1).

Proposition 4.1. [33, Proposition 3.6] *There exists $S > 0$ depending only on $\|(v_0, w_0)\|_{H^{1,m} \times H^{0,m}}$ (the size of the initial data) such that the Cauchy problem (4.3) admits a unique strong solution (v, w) satisfying*

$$(v, w) \in C([0, S]; H^{1,m}(\mathbb{R}^n) \times H^{0,m}(\mathbb{R}^n)).$$

Also, if $(u_0, u_1) \in H^{2,m}(\mathbb{R}^n) \times H^{1,m}(\mathbb{R}^n)$, then the solution (v, w) satisfies

$$(v, w) \in C([0, S]; H^{2,m}(\mathbb{R}^n) \times H^{1,m}(\mathbb{R}^n)) \cap C^1([0, S]; H^{1,m}(\mathbb{R}^n) \times H^{0,m}(\mathbb{R}^n)). \quad (4.5)$$

Moreover, for arbitrarily fixed time $S' > 0$, we can extend the solution to the interval $[0, S']$ by taking ε sufficiently small. Furthermore, if the lifespan

$S_0 = S_0(\varepsilon) = \sup\{S \in (0, \infty); \text{there exists a unique strong solution } (v, w) \text{ to (4.3)}\}$ is finite, then (v, w) satisfies $\lim_{s \rightarrow S_0} \|(v, w)(s)\|_{H^{1,m} \times H^{0,m}} = \infty$.

Next, to obtain an a priori estimate for (v, w) , we decompose (v, w) into the leading terms and the remainder terms. Let $\alpha(s)$ be

$$\alpha(s) = \int_{\mathbb{R}^n} v(s, y) dy, \quad (4.6)$$

which is well-defined due to $v(s) \in H^{1,m}$ with $m > n/2$. We also put

$$\varphi_0(y) = (4\pi)^{-n/2} \exp\left(-\frac{|y|^2}{4}\right).$$

Then, it is easy to see that

$$\int_{\mathbb{R}^n} \varphi_0(y) dy = 1 \quad (4.7)$$

and

$$\Delta_y \varphi_0 = -\frac{y}{2} \cdot \nabla_y \varphi_0 - \frac{n}{2} \varphi_0. \quad (4.8)$$

We also put $\psi_0(y) = \Delta_y \varphi_0(y)$ and decompose v, w as

$$\begin{aligned} v(s, y) &= \alpha(s) \varphi_0(y) + f(s, y), \\ w(s, y) &= \frac{d\alpha}{ds}(s) \varphi_0(y) + \alpha(s) \psi_0(y) + g(s, y), \end{aligned} \quad (4.9)$$

where we expect that (f, g) can be regarded as remainder terms. In order to derive the system that (f, g) satisfies, we first note the following lemma.

Lemma 4.2. [33, Lemma 3.8] *We have*

$$\frac{d\alpha}{ds}(s) = \int_{\mathbb{R}^n} w(s, y) dy, \quad (4.10)$$

$$\frac{e^{-s}}{b(t(s))^2} \frac{d^2\alpha}{ds^2}(s) = \frac{e^{-s}}{b(t(s))^2} \frac{d\alpha}{ds}(s) - \frac{d\alpha}{ds}(s) + \int_{\mathbb{R}^n} r(s, y) dy, \quad (4.11)$$

where r is defined by (4.4).

From the system (4.3), Lemma 4.2 and the equation (4.8), we see that f and g satisfy the following system:

$$\begin{cases} f_s - \frac{y}{2} \cdot \nabla_y f - \frac{n}{2} f = g, & s > 0, y \in \mathbb{R}^n, \\ \frac{e^{-s}}{b(t(s))^2} \left(g_s - \frac{y}{2} \cdot \nabla_y g - \left(\frac{n}{2} + 1 \right) g \right) + g = \Delta_y f + h, & s > 0, y \in \mathbb{R}^n, \\ f(0, y) = v_0(y) - \alpha(0) \varphi_0(y), & y \in \mathbb{R}^n, \\ g(0, y) = w_0(y) - \frac{d\alpha}{ds}(0) \varphi_0(y) - \alpha(0) \psi_0(y), & y \in \mathbb{R}^n, \end{cases} \quad (4.12)$$

where h is given by

$$\begin{aligned} h(s, y) &= \frac{e^{-s}}{b(t(s))^2} \left(-2 \frac{d\alpha}{ds}(s) \psi_0(y) + \alpha(s) \left(\frac{y}{2} \cdot \nabla_y \psi_0(y) + \left(\frac{n}{2} + 1 \right) \psi_0(y) \right) \right) \\ &\quad + r(s, y) - \left(\int_{\mathbb{R}^n} r(s, y) dy \right) \varphi_0(y). \end{aligned} \quad (4.13)$$

Moreover, from (4.6), (4.7) and (4.10), it follows that

$$\int_{\mathbb{R}^n} f(s, y) dy = \int_{\mathbb{R}^n} g(s, y) dy = 0. \quad (4.14)$$

We also notice that the condition (4.14) implies

$$\int_{\mathbb{R}^n} h(s, y) dy = 0. \quad (4.15)$$

4.2. Energy estimates for $n = 1$.

To obtain the decay estimates for f, g , we introduce

$$F(s, y) = \int_{-\infty}^y f(s, z) dz, \quad G(s, y) = \int_{-\infty}^y g(s, z) dz. \quad (4.16)$$

From the following lemma and the condition (4.14), we see that $F, G \in C([0, S]; L^2(\mathbb{R}))$.

Lemma 4.3 (Hardy-type inequality). [33, Lemma 3.9] *Let $f = f(y)$ belong to $H^{0,1}(\mathbb{R})$ and satisfy $\int_{\mathbb{R}} f(y) dy = 0$, and let $F(y) = \int_{-\infty}^y f(z) dz$. Then, we have*

$$\int_{\mathbb{R}} F(y)^2 dy \leq 4 \int_{\mathbb{R}} y^2 f(y)^2 dy. \quad (4.17)$$

Since f and g satisfy the equation (4.12), we can show that F and G satisfy the following system:

$$\begin{cases} F_s - \frac{y}{2} F_y = G, & s > 0, y \in \mathbb{R}, \\ \frac{e^{-s}}{b(t(s))^2} \left(G_s - \frac{y}{2} G_y - G \right) + G = F_{yy} + H, & s > 0, y \in \mathbb{R}, \\ F(0, y) = \int_{-\infty}^y f(0, z) dz, \quad G(0, y) = \int_{-\infty}^y g(0, z) dz, & y \in \mathbb{R}, \end{cases} \quad (4.18)$$

where

$$H(s, y) = \int_{-\infty}^y h(s, z) dz. \quad (4.19)$$

We define the following energy.

$$\begin{aligned} E_0(s) &= \int_{\mathbb{R}} \left(\frac{1}{2} \left(F_y^2 + \frac{e^{-s}}{b(t(s))^2} G^2 \right) + \frac{1}{2} F^2 + \frac{e^{-s}}{b(t(s))^2} F G \right) dy, \\ E_1(s) &= \int_{\mathbb{R}} \left(\frac{1}{2} \left(f_y^2 + \frac{e^{-s}}{b(t(s))^2} g^2 \right) + f^2 + 2 \frac{e^{-s}}{b(t(s))^2} f g \right) dy, \\ E_2(s) &= \int_{\mathbb{R}} y^2 \left[\frac{1}{2} \left(f_y^2 + \frac{e^{-s}}{b(t(s))^2} g^2 \right) + \frac{1}{2} f^2 + \frac{e^{-s}}{b(t(s))^2} f g \right] dy, \\ E_3(s) &= \frac{1}{2} \frac{e^{-s}}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s) \right)^2 + e^{-2\lambda s} \alpha(s)^2, \\ E_4(s) &= \frac{1}{2} \alpha(s)^2 + \frac{e^{-s}}{b(t(s))^2} \alpha(s) \frac{d\alpha}{ds}(s) \end{aligned}$$

and

$$E_5(s) = \sum_{j=0}^4 C_j E_j(s),$$

where λ is a parameter such that $0 < \lambda \leq 1/4$ and C_j ($j = 0, \dots, 4$) are constants such that $C_2 = C_3 = C_4 = 1$ and $1 \ll C_1 \ll C_0$. Then, we have the following energy estimates.

Lemma 4.4. [33, Lemmas 3.10–3.17] *We have*

$$\frac{d}{ds}E_j(s) + \delta_j E_j(s) + L_j(s) = R_j(s),$$

for $j = 0, \dots, 4$, where $\delta_j = \frac{1}{2}$ ($j = 0, 1, 2$), $\delta_3 = 2\lambda$, $\delta_4 = 0$, and

$$\begin{aligned} L_0(s) &= \int_{\mathbb{R}} \left(\frac{1}{2} F_y^2 + G^2 \right) dy, \\ L_1(s) &= \int_{\mathbb{R}} (f_y^2 + g^2) dy - \int_{\mathbb{R}} f^2 dy, \\ L_2(s) &= \int_{\mathbb{R}} y^2 \left(\frac{1}{2} f_y^2 + g^2 \right) dy + 2 \int_{\mathbb{R}} y f_y (f + g) dy, \\ L_3(s) &= \left(\frac{d\alpha}{ds}(s) \right)^2, \\ L_4(s) &= 0 \end{aligned}$$

and

$$\begin{aligned} R_0(s) &= \frac{3}{2} \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}} G^2 dy - \frac{b'(t(s))}{b(t(s))^2} \int_{\mathbb{R}} (G^2 + 2FG) dy + \int_{\mathbb{R}} (F + G) H dy, \\ R_1(s) &= 3 \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}} g^2 dy + 2 \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}} f g dy - \frac{b'(t(s))}{b(t(s))^2} \int_{\mathbb{R}} (g^2 + 4fg) dy \\ &\quad + \int_{\mathbb{R}} (2f + g) h dy, \\ R_2(s) &= \frac{3}{2} \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}} y^2 g^2 dy - \frac{b'(t(s))}{b(t(s))^2} \int_{\mathbb{R}} y^2 (2f + g) g dy + \int_{\mathbb{R}} y^2 (f + g) h dy, \\ R_3(s) &= \frac{1}{2} (2\lambda + 1) \frac{e^{-s}}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s) \right)^2 - \frac{b'(t(s))}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s) \right)^2 \\ &\quad + \frac{d\alpha}{ds}(s) \left(\int_{\mathbb{R}^n} r(s, y) dy \right) + 2e^{-2\lambda s} \alpha(s) \frac{d\alpha}{ds}(s), \\ R_4(s) &= \frac{e^{-s}}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s) \right)^2 - 2 \frac{b'(t(s))}{b(t(s))^2} \alpha(s) \frac{d\alpha}{ds}(s) + \alpha(s) \left(\int_{\mathbb{R}^n} r(s, y) dy \right). \end{aligned}$$

Moreover, we have

$$\frac{d}{ds}E_5(s) + 2\lambda \sum_{j=0}^3 C_j E_j(s) + L_5(s) = R_5(s),$$

where

$$L_5(s) = \sum_{j=0}^2 \left[\left(\frac{1}{2} - 2\lambda \right) C_j E_j(s) + C_j L_j(s) \right] + C_3 L_3(s)$$

and

$$R_5(s) = \sum_{j=0}^4 C_j R_j(s).$$

Furthermore, there exist $C_0 > C_1 > 1$ and $s_0 > 0$ such that

$$\begin{aligned} \|f(s)\|_{H^{1,1}}^2 + \|g(s)\|_{H^{0,1}}^2 + \left(\frac{d\alpha}{ds}(s)\right)^2 &\leq CL_5(s), \\ \|f(s)\|_{H^{1,1}}^2 + \frac{e^{-s}}{b(t(s))^2} \|g(s)\|_{H^{0,1}}^2 + \alpha(s)^2 + \frac{e^{-s}}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s)\right)^2 &\leq CE_5(s) \end{aligned}$$

and

$$|R_5(s)| \leq \frac{1}{2}L_5(s) + Ce^{-\frac{1-\beta}{1+\beta}s}E_5(s) + Ce^{n(p_F-p)s}E_5(s)^p + Ce^{\frac{n}{2}(p_F-p)s}E_5(s)^{\frac{p+1}{2}}$$

are valid for $s \geq s_0$.

4.3. Energy estimates for $n \geq 2$.

When $n \geq 2$, we cannot consider primitives. Instead of them, we define

$$\hat{F}(s, \xi) = |\xi|^{-n/2-\delta} \hat{f}(s, \xi), \quad \hat{G}(s, \xi) = |\xi|^{-n/2-\delta} \hat{g}(s, \xi), \quad \hat{H}(s, \xi) = |\xi|^{-n/2-\delta} \hat{h}(s, \xi),$$

where $0 < \delta < 1$, and $\hat{f}(s, \xi)$ denotes the Fourier transform of $f(s, y)$ with respect to the space variable.

By virtue of the cancelation conditions (4.14), (4.15), $\hat{F}, \hat{G}, \hat{H}$ make sense as L^2 -functions:

Lemma 4.5. [33, Lemma 3.11] *Let $m > n/2+1$ and $f(y) \in H^{0,m}(\mathbb{R}^n)$ be a function satisfying $\hat{f}(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y)dy = 0$. Let $\hat{F}(\xi) = |\xi|^{-n/2-\delta} \hat{f}(\xi)$ with some $0 < \delta < 1$. Then, there exists a constant $C(n, m, \delta) > 0$ such that*

$$\|F\|_{L^2} \leq C(n, m, \delta) \|f\|_{H^{0,m}} \quad (4.20)$$

holds.

We also notice that $\|f\|_{L^2}$ can be controlled by the terms $\|\nabla f\|_{L^2}$ and $\|\nabla F\|_{L^2}$, which come from the diffusion.

Lemma 4.6. [33, (3.39)] *In addition to the assumptions in Lemma 4.5, we further assume $f \in H^1(\mathbb{R}^n)$. Then, for any small $\eta > 0$, there exists a constant $C > 0$ such that we have*

$$\|f\|_{L^2}^2 \leq \eta \|\nabla f\|_{L^2}^2 + C \|\nabla F\|_{L^2}^2$$

holds.

In this case \hat{F} and \hat{G} satisfy the following system.

$$\begin{cases} \hat{F}_s + \frac{\xi}{2} \cdot \nabla_\xi \hat{F} + \frac{1}{2} \left(\frac{n}{2} + \delta \right) \hat{F} = \hat{G}, & s > 0, \xi \in \mathbb{R}^n, \\ \frac{e^{-s}}{b(t(s))^2} \left(\hat{G}_s + \frac{\xi}{2} \cdot \nabla_\xi \hat{G} + \frac{1}{2} \left(\frac{n}{2} + \delta - 2 \right) \hat{G} \right) + \hat{G} = -|\xi|^2 \hat{F} + \hat{H}, & s > 0, \xi \in \mathbb{R}^n. \end{cases}$$

We define the following energy

$$\begin{aligned}
E_0(s) &= \operatorname{Re} \int_{\mathbb{R}^n} \left(\frac{1}{2} \left(|\xi|^2 |\hat{F}|^2 + \frac{e^{-s}}{b(t(s))^2} |\hat{G}|^2 \right) + \frac{1}{2} |\hat{F}|^2 + \frac{e^{-s}}{b(t(s))^2} \hat{F} \bar{\hat{G}} \right) d\xi, \\
E_1(s) &= \int_{\mathbb{R}^n} \left(\frac{1}{2} \left(|\nabla_y f|^2 + \frac{e^{-s}}{b(t(s))^2} g^2 \right) + \left(\frac{n}{4} + 1 \right) \left(\frac{1}{2} f^2 + \frac{e^{-s}}{b(t(s))^2} f g \right) \right) dy, \\
E_2(s) &= \int_{\mathbb{R}^n} |y|^{2m} \left[\frac{1}{2} \left(|\nabla_y f|^2 + \frac{e^{-s}}{b(t(s))^2} g^2 \right) + \frac{1}{2} f^2 + \frac{e^{-s}}{b(t(s))^2} f g \right] dy, \\
E_3(s) &= \frac{1}{2} \frac{e^{-s}}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s) \right)^2 + e^{-2\lambda s} \alpha(s)^2, \\
E_4(s) &= \frac{1}{2} \alpha(s)^2 + \frac{e^{-s}}{b(t(s))^2} \alpha(s) \frac{d\alpha}{ds}(s)
\end{aligned}$$

and

$$E_5(s) = \sum_{j=0}^4 C_j E_j(s),$$

where λ is a parameter such that $0 < \lambda < \min\{\frac{1}{2}, \frac{m}{2} - \frac{n}{4}\}$ and C_j ($j = 0, \dots, 4$) are constants such that $C_2 = C_3 = C_4 = 1$ and $1 \ll C_1 \ll C_0$. Then, we have the following energy estimates.

Lemma 4.7. [33, Lemmas 3.12–3.17] *We have*

$$\frac{d}{ds} E_j(s) + \delta_j E_j(s) + L_j(s) = R_j(s),$$

for $j = 0, \dots, 4$, where $\delta_0 = \delta_1 = \delta$, $\delta_2 = m - \frac{n}{2} - \eta$, $\delta_3 = 2\lambda$, $\delta_4 = 0$, and η is a small parameter such that $0 < \eta < m - \frac{n}{2}$, and

$$\begin{aligned}
L_0(s) &= \frac{1}{2} \int_{\mathbb{R}^n} |\xi|^2 |\hat{F}|^2 d\xi + \int_{\mathbb{R}^n} |\hat{G}|^2 d\xi, \\
L_1(s) &= \frac{1}{2} (1 - \delta) \int_{\mathbb{R}^n} |\nabla_y f|^2 dy + \int_{\mathbb{R}^n} g^2 dy - \left(\frac{n}{4} + \frac{\delta}{2} \right) \left(\frac{n}{4} + 1 \right) \int_{\mathbb{R}^n} f^2 dy, \\
L_2(s) &= \frac{\eta}{2} \int_{\mathbb{R}^n} |y|^{2m} f^2 dy + \frac{1}{2} (\eta + 1) \int_{\mathbb{R}^n} |y|^{2m} |\nabla_y f|^2 dy + \int_{\mathbb{R}^n} |y|^{2m} g^2 dy \\
&\quad + 2m \int_{\mathbb{R}^n} |y|^{2m-2} (y \cdot \nabla_y f) (f + g) dy, \\
L_3(s) &= \left(\frac{d\alpha}{ds}(s) \right)^2, \\
L_4(s) &= 0
\end{aligned}$$

and

$$\begin{aligned}
R_0(s) &= \frac{3}{2} \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}^n} |\hat{G}|^2 d\xi - \frac{b'(t(s))}{b(t(s))^2} \operatorname{Re} \int_{\mathbb{R}^n} (2\hat{F} + \hat{G}) \bar{\hat{G}} d\xi \\
&\quad + \operatorname{Re} \int_{\mathbb{R}^n} (\hat{F} + \hat{G}) \bar{\hat{H}} d\xi, \\
R_1(s) &= \left(\frac{n}{2} + \delta\right) \left(\frac{n}{4} + 1\right) \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}^n} f g dy + \frac{1}{2} (n + 3 + \delta) \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}^n} g^2 dy \\
&\quad - \frac{b'(t(s))}{b(t(s))^2} \int_{\mathbb{R}^n} \left(2 \left(\frac{n}{4} + 1\right) f + g\right) g dy + \int_{\mathbb{R}^n} \left(\left(\frac{n}{4} + 1\right) f + g\right) h dy, \\
R_2(s) &= -\eta \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}^n} |y|^{2m} f g dy - \frac{1}{2} (\eta - 3) \frac{e^{-s}}{b(t(s))^2} \int_{\mathbb{R}^n} |y|^{2m} g^2 dy \\
&\quad - \frac{b'(t(s))}{b(t(s))^2} \int_{\mathbb{R}^n} |y|^{2m} (2f + g) g dy + \int_{\mathbb{R}^n} |y|^{2m} (f + g) h dy, \\
R_3(s) &= \frac{1}{2} (2\lambda + 1) \frac{e^{-s}}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s)\right)^2 - \frac{b'(t(s))}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s)\right)^2 \\
&\quad + \frac{d\alpha}{ds}(s) \left(\int_{\mathbb{R}^n} r(s, y) dy\right) + 2e^{-2\lambda s} \alpha(s) \frac{d\alpha}{ds}(s), \\
R_4(s) &= \frac{e^{-s}}{b(t(s))^2} \left(\frac{d\alpha}{ds}(s)\right)^2 - 2 \frac{b'(t(s))}{b(t(s))^2} \alpha(s) \frac{d\alpha}{ds}(s) + \alpha(s) \left(\int_{\mathbb{R}^n} r(s, y) dy\right).
\end{aligned}$$

Moreover, we have

$$\frac{d}{ds} E_5(s) + 2\lambda \sum_{j=0}^3 C_j E_j(s) + L_5(s) = R_5(s),$$

where

$$\begin{aligned}
L_5(s) &= C_0(\delta - 2\lambda)E_0(s) + C_1(\delta - 2\lambda)E_1(s) + \left(m - \frac{n}{2} - \eta - 2\lambda\right)E_2(s) \\
&\quad + \sum_{j=0}^4 C_j L_j(s)
\end{aligned}$$

and

$$R_5(s) = \sum_{j=0}^4 C_j R_j(s).$$

Furthermore, there exist $C_0 > C_1 > 1$ and $s_0 > 0$ such that

$$\begin{aligned}
\|f(s)\|_{H^{1,m}}^2 + \|g(s)\|_{H^{0,m}}^2 + \left(\frac{d\alpha}{ds}(s)\right)^2 &\leq C L_5(s), \\
\|f(s)\|_{H^{1,m}}^2 + \frac{e^{-s}}{b(t)^2} \|g(s)\|_{H^{0,m}}^2 + \alpha(s)^2 + \frac{e^{-s}}{b(t)^2} \left(\frac{d\alpha}{ds}(s)\right)^2 &\leq C E_5(s)
\end{aligned}$$

and

$$|R_5(s)| \leq \frac{1}{2} L_5(s) + C e^{-\frac{1-\beta}{1+\beta}s} E_5(s) + C e^{n(p_F - p)s} E_5(s)^p + C e^{\frac{n}{2}(p_F - p)s} E_5(s)^{\frac{p+1}{2}}$$

are valid for $s \geq s_0$.

4.4. A priori estimate and the proof of Proposition 1.6.

By Lemmas 4.4 and 4.7 with taking $0 < \lambda < \min\{\frac{1}{2}, \frac{\delta}{2}, \frac{m}{2} - \frac{n}{4}\}$ and η sufficiently small if $n \geq 2$, we can see that (f, g) satisfies the following a priori estimate for $s \geq s_0$. Here we note that the local solution exists for $s > s_0$, provided that ε is sufficiently small by Proposition 4.1.

Lemma 4.8. [33, (3.53)] *There exists $s_0 > 0$ such that for $s \geq s_0$, we have*

$$\frac{d}{ds} E_5(s) \leq C e^{-\frac{1-\beta}{1+\beta}s} E_5(s) + C e^{n(p_F-p)s} E_5(s)^p + C e^{\frac{n}{2}(p_F-p)s} E_5(s)^{\frac{p+1}{2}} \quad (4.21)$$

(where we interpret $1/(1+\beta)$ as an arbitrarily large number when $\beta = -1$).

Now we are in a position to prove Proposition 1.6.

Proof of Proposition 1.6. Let $\varepsilon_1 > 0$ be sufficiently small so that the local solution (v, w) of (4.3) exists for $s > s_0$ (see Proposition 4.1). Therefore, by Lemma 4.8, we see that (f, g) satisfies the a priori estimate (4.21). We put

$$\Lambda(s) := \exp\left(-C \int_{s_0}^s e^{-\frac{1-\beta}{1+\beta}\tau} d\tau\right)$$

(where we interpret $1/(1+\beta)$ as an arbitrarily large number when $\beta = -1$). We note that $c_0 \leq \Lambda(s) \leq 1$ holds for some $c_0 > 0$, and $\Lambda(s_0) = 1$. Multiplying (4.21) by $\Lambda(s)$ and integrating it over $[s_0, s]$, we see that

$$\Lambda(s) E_5(s) \leq E_5(s_0) + C \int_{s_0}^s \left[\Lambda(\tau) e^{n(p_F-p)\tau} E_5(\tau)^p + \Lambda(\tau) e^{\frac{n}{2}(p_F-p)\tau} E_5(\tau)^{\frac{p+1}{2}} \right] d\tau$$

holds for $s_0 \leq s < S_0(\varepsilon)$. Putting

$$M(s) := \sup_{s_0 \leq \tau \leq s} E_5(\tau)$$

and noting

$$M(s_0) \leq C(s_0) \varepsilon^2 \| (a_0, a_1) \|_{H^{1,m} \times H^{0,m}}^2,$$

which can be easily proved by local existence result (see the proof of [33, Proposition 3.5]), we have

$$M(s) \leq C'_0 \varepsilon^2 I_0 + C'_0 \left(e^{n(p_F-p)s} M(s)^p + e^{\frac{n}{2}(p_F-p)s} M(s)^{\frac{p+1}{2}} \right) \quad (4.22)$$

for $s_0 \leq s < S_0(\varepsilon)$ and some $C'_0 > 0$, where $I_0 = \| (a_0, a_1) \|_{H^{1,m} \times H^{0,m}}^2$. Let $S_1 = S_1(\varepsilon) \geq s_0$ is the first time such that M attains the value

$$M(S_1) = 2C'_0 \varepsilon^2 I_0.$$

We note that if $S_0(\varepsilon) = \infty$, then Proposition 1.6 obviously holds, and if $S_0(\varepsilon) < \infty$, then such S_1 actually exists because $\lim_{s \rightarrow S_0(\varepsilon)} M(s) = \infty$. Thus, in what follows we assume $S_0(\varepsilon) < \infty$. Then, substituting $s = S_1$ in (4.22), we see that

$$C'_0 \varepsilon^2 I_0 \leq 2C'_0 \max \left\{ e^{n(p_F-p)S_1} (2C'_0 \varepsilon^2 I_0)^p, e^{\frac{n}{2}(p_F-p)S_1} (2C'_0 \varepsilon^2 I_0)^{\frac{p+1}{2}} \right\}.$$

No matter which quantity attains the maximum, we obtain

$$\varepsilon^{-\frac{2(p-1)}{n(p_F-p)}} \leq C e^{S_1}.$$

Thus, we conclude

$$\varepsilon^{-\frac{1}{\frac{1}{p-1} - \frac{1}{2}}} \leq C (B(T_0(\varepsilon)) + 1).$$

This and the definition of $B(t)$ lead to the desired estimate, and we finish the proof. \square

Proof of Proposition 1.7. In the same way to the derivation of (4.22), noting $p = p_F$, we have

$$M(s) \leq C'_0 \varepsilon^2 I_0 + C'_0 (s - s_0) \left(M(s)^p + M(s)^{\frac{p+1}{2}} \right) \quad (4.23)$$

for $s_0 \leq s < S_0(\varepsilon)$ and some $C'_0 > 0$. Let $S_1 = S_1(\varepsilon) \geq s_0$ is the first time such that M attains the value

$$M(S_1) = 2C'_0 \varepsilon^2 I_0.$$

Moreover, we take $\varepsilon_2 \leq \varepsilon_1$ further small so that $2C'_0 \varepsilon^2 I_0 \leq 1$ holds for $\varepsilon \in (0, \varepsilon_2]$. Then, it is obvious that $M(S_1)^p \leq M(S_1)^{\frac{p+1}{2}}$ for $\varepsilon \in (0, \varepsilon_2]$ and hence, we eventually obtain

$$2C'_0 \varepsilon^2 I_0 \leq C'_0 \varepsilon^2 I_0 + 2C'_0 (S_1 - s_0) (2C'_0 \varepsilon^2 I_0)^{\frac{p+1}{2}}.$$

This implies

$$\exp \left(C \varepsilon^{-(p-1)} + s_0 \right) \leq B(T_0) + 1.$$

Therefore, by the definition of $B(t)$, we have the desired estimate. \square

APPENDIX

Here, we prove existence of a unique strong solution in the sense of Definition 1.1 for the Cauchy problem (1.1).

Proposition 4.9. *Let p satisfy $1 < p < \infty$ ($n = 1, 2$), $1 < p \leq n/(n-2)$ ($n \geq 3$). We assume that $b(t)$ is a smooth nonnegative function. Let $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then, there exist a constant $T > 0$ and a unique strong solution u of the Cauchy problem (1.1) on $[0, T)$. Moreover, we have the blow-up alternative, that is, for the lifespan T_0 defined in Definition 1.2, if $T_0 < \infty$, then*

$$\lim_{t \rightarrow T_0^-} \|(u, u_t)(t)\|_{H^1 \times L^2} = \infty.$$

Proof. Let $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and let $F \in L^1_{\text{loc}}([0, \infty); L^2(\mathbb{R}^n))$. First, we recall that the strong solution of the linear problem

$$\begin{cases} \square u + b(t)u_t = F, & t > 0, x \in \mathbb{R}^n, \\ u(0) = u_0, \quad u_t(0) = u_1, & x \in \mathbb{R}^n \end{cases} \quad (4.24)$$

can be easily obtained via the Fourier transform and the Duhamel principle, and the corresponding solution belongs to $C^1([0, T); L^2(\mathbb{R}^n)) \cap C([0, T); H^1(\mathbb{R}^n))$. Moreover, by an approximation argument, we see that the solution u satisfies the energy identity

$$\begin{aligned} & \frac{1}{2} \|(\nabla_x u, u_t)(t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^n} b(\tau) u_t(\tau, x)^2 dx d\tau \\ &= \frac{1}{2} \|(\nabla_x u_0, u_1)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^n} F(\tau, x) u_t(\tau, x) dx d\tau \end{aligned}$$

for $t > 0$. Combining this with a Gronwall-type inequality (see [32, Lemma 9.12]) and the assumption $b(t) \geq 0$, we further obtain

$$\|(\nabla_x u, u_t)(t)\|_{L^2} \leq \|(\nabla_x u_0, u_1)\|_{L^2} + \int_0^t \|F(\tau)\|_{L^2} dx d\tau \quad (4.25)$$

for $t > 0$. To construct a solution of the Cauchy problem (1.1), we employ the contraction mapping principle. To this end, we take a constant $R > 0$ such that $\|(u_0, u_1)\|_{H^1 \times L^2} \leq R$ and define

$$K(T, R) := \left\{ v \in C^1([0, T]; L^2(\mathbb{R}^n)) \cap C([0, T]; H^1(\mathbb{R}^n)); \sup_{t \in [0, T]} \|(v, v_t)(t)\|_{H^1 \times L^2} \leq 3R \right\}.$$

We define the metric

$$d(u, v) := \sup_{t \in [0, T]} \|(u - v, u_t - v_t)(t)\|_{H^1 \times L^2}$$

for $u, v \in K(T, R)$. Then, $K(T, R)$ is a complete metric space with the metric $d(\cdot, \cdot)$. Let $1 < p < \infty$ ($n = 1, 2$), $1 < p \leq n/(n-2)$ $n \geq 3$. Let $u^{(0)}$ be the solution of the linear problem (4.24) with $F = 0$. Then, for $j = 1, 2, \dots$, we successively define $u^{(j)}$ as the solution of the linear problem (4.24) with $F = |u^{(j-1)}|^p$. By (4.25), the first approximation $u^{(0)}$ clearly belongs to $K(T, R)$. Furthermore, by the Sobolev embedding theorem, we see that if $u^{(j-1)} \in K(T, R)$, then

$$\begin{aligned} \|(\nabla u^{(j)}, u_t^{(j)})(t)\|_{L^2} &\leq R + \int_0^t \|u^{(j-1)}(\tau)\|_{L^{2p}}^p d\tau \\ &\leq R + C \int_0^t \|u^{(j-1)}(\tau)\|_{H^1}^p d\tau \\ &\leq R + CTR^p. \end{aligned}$$

We also estimate

$$\begin{aligned} \|u^{(j)}(t)\|_{L^2} &\leq \|u_0\|_{L^2} + \int_0^t \|u_t^{(j)}(\tau)\|_{L^2} d\tau \\ &\leq R + T(R + CTR^p). \end{aligned}$$

Therefore, taking $T > 0$ sufficiently small so that $CTR^p + T(R + CTR^p) \leq R$ holds, we obtain $u^{(j)} \in K(T, R)$. Thus, it follows from the mathematical induction that $u^{(j)} \in K(T, R)$ for all $j \geq 0$. Next, making use of

$$|u|^p - |v|^p \leq C(|u| + |v|)^{p-1}|u - v|,$$

we estimate

$$\begin{aligned} &\|(\nabla u^{(j)}(t) - \nabla u^{(j-1)}(t), u_t^{(j)}(t) - u_t^{(j-1)}(t))\|_{L^2} \\ &\leq C \int_0^t \|(|u^{(j-1)}(\tau)| + |u^{(j-2)}(\tau)|)^{p-1} |u^{(j-1)}(\tau) - u^{(j-2)}(\tau)|\|_{L^2} d\tau \\ &\leq C \int_0^t \|(|u^{(j-1)}(\tau)| + |u^{(j-2)}(\tau)|)\|_{L^{2p}}^{p-1} \|u^{(j-1)}(\tau) - u^{(j-2)}(\tau)\|_{L^{2p}} d\tau \\ &\leq C \int_0^t (\|u^{(j-1)}(\tau)\|_{H^1} + \|u^{(j-2)}(\tau)\|_{H^1})^{p-1} \|u^{(j-1)}(\tau) - u^{(j-2)}(\tau)\|_{H^1} d\tau \\ &\leq CTR^{p-1} d(u^{(j-1)}, u^{(j-2)}). \end{aligned}$$

We also have

$$\begin{aligned} \|u^{(j)}(t) - u^{(j-1)}(t)\|_{L^2} &\leq \int_0^t \|u_t^{(j)}(\tau) - u_t^{(j-1)}(\tau)\|_{L^2} d\tau \\ &\leq CT^2 R^{p-1} d(u^{(j-1)}, u^{(j-2)}). \end{aligned}$$

Therefore, taking $T > 0$ further small so that $CTR^{p-1} + CT^2R^{p-1} \leq \kappa$ with some $\kappa \in (0, 1)$, we conclude

$$d(u^{(j)}, u^{(j-1)}) \leq \kappa d(u^{(j-1)}, u^{(j-2)}).$$

Hence, $\{u^{(j)}\}_{j \geq 0}$ is a Cauchy sequence in $K(T, R)$ and we find the limit $u \in K(T, R)$. Since each $u^{(j)}$ satisfies the initial conditions $u^{(j)}(0) = u_0$ and $u_t^{(j)}(0) = u_1$, so does u . Moreover, taking the limit $j \rightarrow \infty$ in the equation

$$u_{tt}^{(j)} = \Delta u^{(j)} - b(t)u_t^{(j)} + |u^{(j)}|^p,$$

which is valid in $C([0, T]; H^{-1}(\mathbb{R}^n))$, we see that $u \in C^2([0, T]; H^{-1}(\mathbb{R}^n))$ holds and u is a strong solution of (1.1). Indeed, let $t_0 \in [0, T)$ and take a small neighborhood ω of t_0 in $[0, T)$. Then, we have

$$\Delta u^{(j)} \rightarrow \Delta u, \quad b(t)u_t^{(j)} \rightarrow b(t)u_t, \quad |u^{(j)}|^p \rightarrow |u|^p$$

in $C(\omega; H^{-1}(\mathbb{R}^n))$ as $j \rightarrow \infty$, and this convergence is uniform in ω . Thus, there exists $w \in C(\omega; H^{-1}(\mathbb{R}^n))$ such that $u_{tt}^{(j)} \rightarrow w$ in $C(\omega; H^{-1}(\mathbb{R}^n))$ as $j \rightarrow \infty$, and this convergence is also uniform in ω . Thus, we compute

$$\begin{aligned} &\lim_{h \rightarrow 0} \left\| \frac{1}{h} (u_t(t_0 + h) - u_t(t_0)) - w(t_0) \right\|_{H^{-1}} \\ &= \lim_{h \rightarrow 0} \lim_{j \rightarrow \infty} \left\| \frac{1}{h} (u_t^{(j)}(t_0 + h) - u_t^{(j)}(t_0)) - u_{tt}^{(j)}(t_0) \right\|_{H^{-1}} \\ &= \lim_{j \rightarrow \infty} \lim_{h \rightarrow 0} \left\| \frac{1}{h} (u_t^{(j)}(t_0 + h) - u_t^{(j)}(t_0)) - u_{tt}^{(j)}(t_0) \right\|_{H^{-1}} \\ &= 0, \end{aligned}$$

which leads to $u_{tt} = w$ and hence, $u \in C^2([0, T]; H^{-1}(\mathbb{R}^n))$.

In order to show the uniqueness, let u, v are strong solutions of (1.1) on $[0, T)$ with the initial data (u_0, u_1) . Let $M = \sup_{t \in [0, T)} (\|u(t), u_t(t)\|_{H^1 \times L^2} + \|(v(t), v_t(t))\|_{H^1 \times L^2})$. Then, a similar argument above implies

$$\begin{aligned} &\|(u(t) - v(t), u_t(t) - v_t(t))\|_{H^1 \times L^2} \\ &\leq (1 + CM^{p-1}) \int_0^t \|(u(\tau) - v(\tau), u_t(\tau) - v_t(\tau))\|_{H^1 \times L^2} d\tau. \end{aligned}$$

This and the Gronwall inequality give $u = v$.

Finally, we prove the blow-up alternative. Let the lifespan T_0 be finite and we suppose that

$$\lim_{t \rightarrow T_0^-} \|(u, u_t)(t)\|_{H^1 \times L^2} < \infty.$$

Then, there exists a constant $R' > 0$ such that $\|(u, u_t)(t)\|_{H^1 \times L^2} \leq R'$ for any $t \in [0, T_0)$. Let $t_0 \in [0, T_0)$ be arbitrarily fixed. In the same way as before, we see that there exists $T' > 0$ depending only on R' such that we can construct a unique

strong solution on $[t_0, t_0 + T']$. However, if $t_0 \in (T_0 - T', T_0)$, this contradicts the definition of the lifespan. \square

ACKNOWLEDGMENTS

The authors are deeply grateful to Professor Mitsuru Sugimoto for his helpful comments. The first, second and third authors were partly supported by the Japan Society for the Promotion of Science, Grant-in-Aid for JSPS Fellows No. 16J30008, 14J01884 and 15J01600, respectively.

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